Lecture 20. The dimension of a vector space

<u>Thm</u> Given a vector space V, every basis of V has the same number of vectors.

<u>Def</u> Given a vector space V, its <u>dimension</u> $\dim(V)$ is defined as the number of vectors in a basis.

e.g. $\dim(\mathbb{R}^n) = n$ (basis : $\vec{e_1}, \vec{e_2}, \cdots, \vec{e_n}$)

$\dim(\mathbb{P}_n) = n+1$ (basis : $1, t, \cdots, t^n$)

<u>Note</u> We can classify subspaces of $\mathbb{R}^n$ by their dimensions

e.g. dimension 0 : zero space (consisting only of $\vec{0}$)

dimension 1 : a line through the origin (spanned by 1 vector)

dimension 2 : a plane through the origin

(spanned by 2 linearly independent vectors)

<u>Def</u> A vector space is <u>infinite dimensional</u> if it has no finite bases.

e.g. • the space of all polynomials (basis : $1, t, t^2, \cdots$)

• the space of all continuous functions $f : \mathbb{R} \longrightarrow \mathbb{R}$

(basis impossible to write down)

<u>Note</u> In Math 313, we will not discuss much about infinite dimensional vector spaces.

Def Let A be a matrix.

(1) Its *nullity* is the dimension of $\text{Nul}(A)$.

(2) Its *rank* is the dimension of $\text{Col}(A)$.

Thm (Rank-nullity theorem)

Let A be a matrix.

(1) Its nullity counts the columns in RREF(A) without a leading 1.

(2) Its rank counts the columns in RREF(A) with a leading 1.

(3) The sum of its rank and nullity counts the columns in A.

pf (1) A basis of $\text{Nul}(A)$ is given by parametrizing the solution of the equation $A\vec{x} = \vec{0}$

$\Rightarrow$ $\dim(\text{Nul}(A))$ counts the free variables of the equation $A\vec{x} = \vec{0}$

$\Rightarrow$ $\dim(\text{Nul}(A))$ counts the columns in RREF(A) without leading 1s

(2) A basis of $\text{Col}(A)$ is given by the columns that contain a position of a leading 1 in RREF(A).

$\Rightarrow$ $\dim(\text{Col}(A))$ counts the columns in RREF(A) with leading 1s

(3) By (1) and (2), the rank and nullity of A together count the columns in RREF(A), or equivalently the columns in A.

Note If we can identify a vector space as a column space or a null space, we can compute its dimension using the rank-nullity theorem.

Ex Find the rank and nullity of each matrix.

(1) $A = \begin{bmatrix} 0 & 1 & 2 \\ 3 & 4 & 5 \end{bmatrix}$

Sol We find

$$\text{RREF}(A) = \begin{bmatrix} \textcircled{1} & 0 & -1 \\ 0 & \textcircled{1} & 2 \end{bmatrix}.$$

(2 columns with a leading 1 and 1 column without a leading 1)

$\Rightarrow$ A has rank 2 and nullity 1

(2) $B = \begin{bmatrix} 3 & 2 & 4 \\ 1 & 5 & 0 \\ 4 & 0 & 1 \end{bmatrix}$

Sol We find

$$\text{RREF}(B) = \begin{bmatrix} \textcircled{1} & 0 & 0 \\ 0 & \textcircled{1} & 0 \\ 0 & 0 & \textcircled{1} \end{bmatrix}$$

(3 columns with a leading 1 and 0 columns without a leading 1)

$\Rightarrow$ B has rank 3 and nullity 0

(3) $C = \begin{bmatrix} 2 & 3 & 7 & 8 \\ 3 & -4 & 2 & -5 \\ 0 & 1 & 1 & 2 \end{bmatrix}$

Sol We find

$$\text{RREF}(C) = \begin{bmatrix} \textcircled{1} & 0 & 2 & 1 \\ 0 & \textcircled{1} & 1 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

(2 columns with a leading 1 and 2 columns without a leading 1)

$\Rightarrow$ C has rank 2 and nullity 2

Ex Find the dimension of each vector space.

(1) The set of all points (x,y,z) in $\mathbb{R}^3$ with $y=3x+4z$ and $4x=3y+3z$

Sol The set is given by the solutions of the linear system

$$\begin{cases} 3x - y + 4z = 0 \\ 4x - 3y - 3z = 0 \end{cases}$$

Hence we can identify the set as the null space of

$$A = \begin{bmatrix} 3 & -1 & 4 \\ 4 & -3 & -3 \end{bmatrix} \text{ with } \mathrm{RREF}(A) = \begin{bmatrix} ① & 0 & 3 \\ 0 & ① & 5 \end{bmatrix}.$$

$\Rightarrow \dim(\mathrm{Nul}(A)) = \boxed{1}$ (number of columns without a leading 1)

(2) The set of all vectors of the form

$$\begin{bmatrix} 3a-2b \\ a+2c \\ b+3c \end{bmatrix} \text{ with } a, b, c \in \mathbb{R}.$$

Sol We may write $\begin{bmatrix} 3a-2b \\ a+2c \\ b+3c \end{bmatrix} = a\begin{bmatrix} 3 \\ 1 \\ 0 \end{bmatrix} + b\begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix} + c\begin{bmatrix} 0 \\ 2 \\ 3 \end{bmatrix}$

Hence we can identify the set as the column space of

$$A = \begin{bmatrix} 3 & -2 & 0 \\ 1 & 0 & 2 \\ 0 & 1 & 3 \end{bmatrix} \text{ with } \mathrm{RREF}(A) = \begin{bmatrix} ① & 0 & 2 \\ 0 & ① & 3 \\ 0 & 0 & 0 \end{bmatrix}.$$

$\Rightarrow \dim(\mathrm{Col}(A)) = \boxed{2}$ (number of columns with a leading 1)

(3) The set of all polynomials $p(t) \in \mathbb{P}_2$ with roots at $t = 2, 3$

Sol Consider the linear transformation $T : \mathbb{P}_2 \longrightarrow \mathbb{R}^2$ given by

$$T(p(t)) = \begin{bmatrix} p(2) \\ p(3) \end{bmatrix}.$$

The set is given by the solutions of the equation $T(p(t)) = \vec{0}$

The standard matrix has columns $T(1), T(t), T(t^2)$.

$$p(t) = 1 : \ p(2) = 1, \ p(3) = 1 \Rightarrow T(1) = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$p(t) = t : \ p(2) = 2, \ p(3) = 3 \Rightarrow T(t) = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$$

$$p(t) = t^2 : \ p(2) = 4, \ p(3) = 9 \Rightarrow T(t^2) = \begin{bmatrix} 4 \\ 9 \end{bmatrix}$$

Hence the standard matrix is

$$A = \begin{bmatrix} 1 & 2 & 4 \\ 1 & 3 & 9 \end{bmatrix} \text{ with } \mathrm{RREF}(A) = \begin{bmatrix} \textcircled{1} & 0 & -6 \\ 0 & \textcircled{1} & 5 \end{bmatrix}.$$

Now we can convert the equation $T(p(t)) = \vec{0}$ into a matrix equation $A\vec{x} = \vec{0}$ by setting $\vec{x} = [p(t)]$.

Then we can identify the set as $\mathrm{Nul}(A)$.

$\Rightarrow \dim(\mathrm{Nul}(A)) = \boxed{1}$ (number of columns without a leading 1)

Note In fact, if $p(t)$ has roots at $t = 2, 3$ with degree at most 2, we have $p(t) = c\underbrace{(t-2)(t-3)}_{\text{basis}}$ with $c \in \mathbb{R}$.